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Identification of Cubic Stiffness Nonlinearity by Linearity -Conserved NARMAX Modeling

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A modeling technique to estimate a NARMAX model is developed to identify nonlinearities which are contained in linear-based nonlinear systems. Considering great contributions by linear parts of the NARMAX model on describing nonlinearities, a linear model, which is estimated from small amplitude input and the corresponding output is taken as the linear part of the NARMAX model. Hence, the capabilities of the model to predict nonlinear behaviors for any input within stable region are fairly improved, and multiplicity problem in selecting a nonlinear regression model is also resolved. As an illustration, one degree of freedom system with cubic stiffness is identified in terms of NARMAX modeling technique using the procedure proposed in this work and conventional one, respectively. By extraction higher order FRFs from the NARMAX models, dominant nonlinearities of the system are predicted, and the results by the two methods are compared with analytic one, which shows the priority of the modeling technique proposed.

Key Words : NARMAX(Nonlinear Auto-Regressive Moving Average with eXogenous input) Model, Multiplicity, Two Step Modeling, Linear Property, Generalized FRF (Frequency Response Function)

Nomenclature

$\hat{}$	: Estimated value
AIC	: Akaike's information criterion
ERR_i	: Error reduction ratio of the i-th model term
FRF	: Frequency response function
$H_n(f_1, \cdots, f_n)$	: nth order frequency response function
N	: Number of data in the processing
$NARMAX$	: Nonlinear Auto-Regressive Moving Averge with eXogenous input
$e(t)$	: Residual Sequence
f_i	: i th frequency component in generalized frequency response function
n	: degree of nonlinearity
p, q, r	: order of output, input and residual sequence
$u_t(=u(t))$	: input sequence
$y_t(=y(t))$	: output sequence
$\alpha_i(t)$	: ith term in NARMAX model
θ_i	: ith NARMAX model parameter

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1. Introduction

Time series modeling has been applied in many fields mainly for two purposes. One is for control of a system, in which case the prime object is to predict the system output accurately by minimizing the residuals. The other is for identification of a system, in which case the object is to give good descriptions of the physical properties of the system. Recently nonlinear systems as well as linear systems have been tackled by this technique intensively in such a way that both of the two purposes can be satisfied(Billings and Tsang, 1989a, b). Most of the literatures so far published, however, are oriented to the system control rather than system identification.

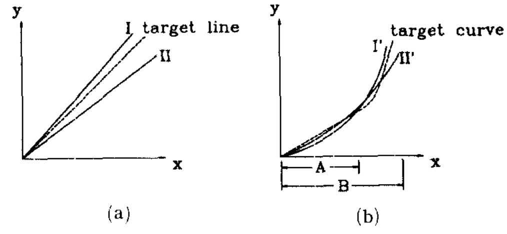
Wide class of nonlinearities in mechanical systems are assumed to exist as restoring forces, they are often expressed as polynomial expansions of a displacement and/or a velocity of a concentrated mass. Mechanical elements such as a rubber spring, a disk spring, a vlolute spring, etc. and an assembled unit of a loudspeaker are good examples(Wall, 1963 ; Kaizer, 1987). The system with these kinds of elements behaves like a linear system for small amplitude inputs. Increasing the amplitude, nonlinearities such as harmonics and gain variations appear. Polynomial NARMAX modeling(Billings and Tsang, 1989a ; Chen and Billings, 1989a) may be a strong tool to identify these types of nonlinear systems because of their similarity in polynomial expansions. The NARMAX modeling is recently accepted as one of the most practical technique for nonlinearity identifications because the experimental setup and procedure required are not different from the linear case. Furthermore, once an adequate model is fitted, sufficient nonlinearity information can be extracted from the model. As one could expect, however, there are several obstacles to resolve before getting an adequate model, which are determination of model order, nonlinearity order, model structure, etc.

Nonlinear relationship between the input and the output sequence gives much richer possibilities to describe systems, each of them has its own characteristics respectively. Therefore, it becomes necessary to select the model which best fits the aim of an analysist. In this work the aim of the time series modeling is to identify nonlinear system dynamics such as harmonics, intermodulation, gain variation, etc. To achieve the pupose, a NARMAX modeling technique is proposed, which conserve the linear properties of the system in the nonlinear time series model. At first, a multiplicity problem in nonlinear regressions of nonlinear relations between the input and output is discussed, and the effects of the linear system properities on descriptions of its nonlinear behaviors are clarified. A nonlinear ARMAX modeling technique based on the linear terms is suggested in section 3. In the method, the modeling is made by two steps, which are linear modeling for small amplitude input by a linear ARMAX model and nonlinear modeling for a large amplitude input based on the results obtained at the linear modeling. The method is applied to a single dof system with a hardening stiffness to show its feasibilities, where nonlinearities of the system are analyzed in terms of generalized FRFs.

2. Multiplicity of Nonlinear Regression Models

Unqueness of a regression model in linear system theory is well defined(Söderström and Stoica, 1989). By taking an unbiased estimator among several ones available and choosing an adequate model order, the model can be easily estimated so as to satisfy the whiteness of the residuals and the uniqueness. As a linear regression model is composed of specified variables only such as $y(t)$, $u(t)$ and $e(t)$, it is easy to build the adequate model. From figure 1, which conceptually illustrate linear regressions of linear relation between the input and output and nonlinear regressions of nonlinear one, respectively, it is easy to judge, in the viewpoint of minimizing errors, that the linear model I in Fig. 1(a) is better than the model II. The model I, moreover, is necessarily expected to represent well the system characteristics such as system poles and residues.

In case of a nonlinear regression of unknown nonlinear relation between the input and output, it's quite an another story because of the multi-



(a) Line fitting by linear regression models
(b) Curve fitting by nonlinear regression models

Fig. 1 Illustration of multiplicity in selecting nonlinear regression models plicity in selecting a model. Two nolinear models are suggested in Fig. 1(b) in order to fit the target curve(dashed line), respectively. As the nonlinear model, in general, has amplitude-dependent characteristics, the input range was specified as A or B in the figure. Let's assume that the residual sum of squares by the two models are the same and both two models are unbiased. In this case, it is not clear that which is the better, and therfore, the judgement is wholly up to an analyst. As far as only the statistics are concerned, either of two models I' and II' will be satisfactory. If we give weights on the lower half range of the input (range A), the model II' will be preferred.

To select a model among multiple ones, it is necessary first to define the intended use of the model before determining the way how to model the system(Söderström and Stoica, 1989). The final purpose of the modeling in this paper is to predict various nonlinear behaviors of the system for any input within stable region in the frequency domain as well as in the amplitude domain. The concrete procedure to estimate the model which fits the aim is presented in the following section.

3. NARMAX Modeling Based on Linear Characteristics

Consider a nonlinear system which is expressed by the following equation :

$$m\ddot{y}+c\dot{y}+ky+g(\dot{y},\ y)=u \tag{1}$$

where m, c and k are mass and linear damping and spring coefficient respectively and the g is some polynomial function of $\dot{y}$ and y. The system behaves like a linear system for sufficiently small amplitude of y(or u) because the contributions by the function g is negligible compared with those by the linear terms. When the input amplitude increases to some extent, the amplitudes of the output $\dot{y}$ and y also increase and the contributions by the function g became not negligible. Wide class of mechanical systems under practical uses follow the above relationship, which may be classified as a linear-based nonlinear system. As the degree of nonlinearity of them are generally assumed to be weak, the linear characteristics are still predominant even in nonlinear responses.

Gifford(1993) derived the nonlinear FRFs, as a special case, for a cubic stiffness nonlinear system as Eq. (2) by applying the concept of Volterra series(Rugh, 1981 ; Gifford and Tomlinson, 1989), which are listed in Eq. (3).

$$m\ddot{y}+c\dot{y}+ky+k'y^3=u \tag{2}$$

$$\begin{aligned} H_1(j\omega_1) &= \frac{1}{k-m\omega_1^2+jc\omega_1} \\ H_3(j\omega_1,\ j\omega_2,\ j\omega_3) &= -\frac{k'}{m}\cdot H_1(j\omega_1)\cdot H_1(j\omega_2) \\ &\quad \cdot H_1(j\omega_3) \\ &\quad \cdot H_1(j\omega_1+j\omega_2+j\omega_3) \end{aligned} \tag{3}$$

There does not exist an FRF of even order. It is because that the system (2) has no variables of even order and the response for the system is symmetrical. As shown in Eq. (3), we can see that the nonlinear FRF of order 3 is expressed by multiplications of the linear FRF only, which clearly shows that accurate descriptions of nonlinear behaviors are desperately dependent upon whether the linear FRF is properly derived or not.

As an another way of deriving the higher order FRFs, there is a probing method(Bedrosian and Rice, 1971). As shown in Fig. 2, which illustrates a schematic relation between the NARMAX model terms and the generalized FRFs, the 1st order FRF is described completely by the linear terms only and the nth order FRF by all of terms up to the order n. The coefficients of linear auto-regressive part are always used as a denominator of the FRFs regardless of the order. From the generalized FRFs, various nonlinearities can be described for the inputs of specified amplitude

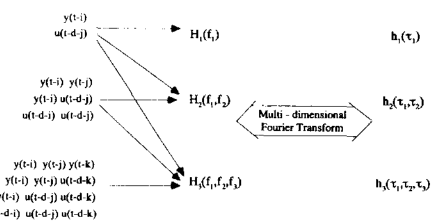


Fig. 2 Schemetic relation between generalized FRFs and NARMAX model terms

Table 1 Approximate equations for estimating various nonlinear responses for given inputs

	Input	Output
Harmonics	$u(t)=A\cos(2\pi ft)$	2nd harmonic : $\lvert E_2\rvert \approx \frac{\lvert A\rvert^2}{2}\lvert H_2(f,f)\rvert$ 3rd harmonic : $\lvert E_3\rvert \approx \frac{\lvert A\rvert^3}{2^2}\lvert H_3(f,f,f)\rvert$ m-th harmonic : $\lvert E_m\rvert \approx \frac{\lvert A\rvert^m}{2^{m-1}}\lvert H_m(f,\cdots,f)\rvert$
Gain Variation	$u(t)=A\cos(2\pi ft)$	$y(t)=\lvert N(A,f)\rvert\cos(2\pi ft+\angle N(A,f))$ $\lvert N(A,f)\rvert \approx A\cdot\lvert H_1(f)\rvert\cdot\left\lvert 1+\frac{3}{4}\lvert A\rvert^2\frac{H_3(f,-f,f)}{H_1(f)}\right\rvert$
Intermodulation	$u(t)=A\cos(2\pi ft)$ $+B\cos(2\pi gt)$	$Mag(f+g)=\lvert A\cdot B\cdot H_2(f,g)\rvert$ $Mag(f-g)=\lvert A\cdot B\cdot H_2(f,-g)\rvert$, $(f>g)$

and frequency, some of which are listed in Table 1(Billings and Tsang, 1989b). This also shows how important the determination of the linear part of an NARMAX model is.

Because of the relation between the NARMAX model and the higher order FRFs, although addition of one nonlinear term in the NARMAX modeling may cause little contribution to the reduction of the prediction errors, it may bring about the change of most of the model parameters and consequently all the FRFs of the order from 1 to n. Therefore, it is necessary to establish a modeling strategy in which the physical properties of the system are less sensitive to the nonlinear model parameters. In this paper, a NARMAX model is obtained for the systems expressed as Eq. (1) by putting an emphasis on the dynamics of the system in the linear range(Jang, 1994). The modeling is made by two steps. The first step is the linear modeling for the low level input and the corresponding output, and the second step is nonlinear modeling for large amplitude input and the output. The concrete procedure is stated in the following section.

3.1 Determination of linear model in the linear range

For a set of stationary input and ouput sequence of a nonlinear time-invariant system, if they are bounded to some level, it is possible to estimate a NARMAX model which satisfy a stationarity condition within the specified region and does not diverge out(Chen and Billings, 1989b). This model is, at least, expected to be Valid for the input and output, of which amplitudes are less than those used at nonlinear modeling. In this case, each term of the NARMAX model has specified significance to the system response. When the input amplitude decreases to small one so as to neglect the nonlinearities, contribution of the nonlinear terms to the system responses approaches zero first because their high sensitivity to amplitude. It is natural, therefore, that the linear part of the model is responsible for the linear characteristics like the case of the system expressed by Eq. (1).

Considering the great contribution of system dynamics as discussed previously, the linear modeling is executed first to obtain the linear model and the corresponding linear characteristics. A small amplitude input and the corresponding output are used in this modeling so as to neglect the nonlinearity of the system. An adequate order or maximum time lag is determined by AIC. The model parameters are estimated by using the generalized least squares algorithm, which determines the error model simultaneously(Sinha and Kuszta, 1983; Korenberg et al., 1986).

The linear model structure estimated will be used as the linear part of a nonlinear model. Extending the model order obtained from the linear modeling to nonlinear modeling is natural because the model order for the mechanical system is not affected by input level but by the degree of freedoms of the system or the maximum derivative order about the concentrated mass. Some examples show that the NARMAX model with maximum time lag 2 can represent well the system of single degree of freedom(Seidel and Davies, 1988 ; Tsang and Billings, 1992).

3.2 Regression of filtered responses using nonlinear variables

The NARMAX model of Eq. (4) can be rewritten as a linear form in the parameter model such as Eq. (5) :

$$y(t)=F^n\{y(t-1), \cdots, y(t-p), u(t-d), \cdots, u(t-d-q), e(t-1), \cdots, e(t-r)\}+e(t) \quad (4)$$

$$y(t)=\sum_{h=0}^{\#\text{ of model terms}} \theta_h \alpha_h(t)+e(t) \quad (5)$$

where y, u, e mean the measured output, input and the prediction error, respectively and p, q, r the corresponding orders, and F^n is some polynomial-type nonlinear function of maximum degree n. The α's mean multiplications of the y, u and e, respectively as Eq. (6), and θ's the corresponding coefficients :

$$\alpha_0(t)=1$$

$$\alpha_h(t)=\prod_{i=1}^{p} y^{n_i}(t-i)\cdot\prod_{i=1}^{q} u^{n_j}(t-d-j) \cdot\prod_{i=1}^{r} e^{n_k}(t-k) \quad \text{for } h\geq 1 \quad (6)$$

where

n_i, n_j, n_k : nonlinearity order of y, u and e

$\sum_{i,j,k}(n_i+n_j+n_k)=1, \cdots, n$

$h=h(p, q, r, n_i, n_j, n_k)$

Eq. (5) can be rewritten as

$$y(t)=\sum_{h=0}^{\#\text{ of linear terms}} \theta_{L,h}\alpha_{L,h}(t)+\sum_{h=0}^{\#\text{ of nonlinear terms}} \theta_{N,h}\alpha_{N,h}(t)+e(t) \quad (7)$$

where subscripts L and N mean the linear and the nonlinear part of the NARMAX model respecitively. The first summation of the right side of Eq. (7) takes the form of a conventional ARMAX model and the second part is responsible for regressions by nolinear variables only.

As a solution to conserve the linear characteristics in the NARMAX model, linear model coefficients $\theta_{L,h}$'s are fixed by those estimated at the linear modeling of the first step. The output responses are filtered by the linear model coefficients as Eq. (8), then a residual sequence $z(t)$ are regressed again with nonlinear variables only as Eq. (9) including an error model.

$$z(t)=y(t)-\sum_{h=0}^{\#\text{ of linear terms}} \theta_{L,h}\alpha_{L,h}(t) \quad (8)$$

$$z(t)=\sum_{h=0}^{\#\text{ of nonlinear terms}} \theta_{N,h}\alpha_{N,h}(t)+e(t) \quad (9)$$

The orthogonalization algorithm is used to avoid a computational burden, and the judgement that which term should be included into the model is made by using an error reduction ratio(Korengerg and et al., 1986). As a final step, the estimated nonlinear model is validated by the correlation test(Billings and Voon, 1986).

The above procedure gives constraints to the estimation of nonlinear coefficients as well as linear ones, and therefore, the model set may deviate from a global minimum point regardless that the model satisfies the condition of unbiasness. But the choice of model should be done so as to fit the intended use. In this paper the object of the model is to identify the nonlinear system, in other words, to describe accurately various nonlinear behaviors within the applicable input range.

When applying this technique, in which the linear model estimated from the linear range is used as the linear part of the nonlinear model, it should be preceded to judge whether the linear model represent well the linear characteristics or not. In order to obtain the linear model, small amplitude input must be selected so as to guarantee that the nonlinearities are negligible and the system behaves linearly around at the input amplitude. It is easily confirmed through the re-estimation of the system model for another input, of which amplitude is slightly different from that of the input. If the change of the model coefficients is negligible or falls within their confidence interval, it will be concluded that the linear model represent the linear dynamics of the system well. In the circumstances that nonlinear effects can not be excluded even for a low level input, it may be dangerous to take a linear model as the linear part of an NARMAK model, and another method to conserve the linear characteristics of the system in the time series model is required(Jang, 1994 ; Jang and Kim, 1994).

4. Simulation on Identifications of Hardening Stiffness Model

A single degree of freedom system with cubic stiffness as Eq. (2) is considered for an illustration, where m, c, k and k' are given by 1 kg, 2 N•s/m, 60 N/m and 60 N/m^3 respectively. The natural frequency of the system is 1.233 Hz when the effect of the nonlinear term is neglected. The system was excited by a band-passed(0~2 Hz) uniformly distributed random signal with zero mean. To take an advantage of easy controllability of a maximum amplitude for a nonlinear system which has amplitude-dependent characteristics, a uniformly distributed random input was used here rather a Gaussian random input(Billings and Voon, 1984). Two levels of inputs with root mean square (rms) values of 0. 254 N and 146 N were used. The corresponding rms levels of the outputs are 0.0086 m and 1.1 m, respectively. In consideration of the effects by the harmonics and inter-modulations, the sampling interval was chosen as 50(msec) so that Nyquist cut-off frequency is 10 Hz. The time series model was constructed from 900 data points of input and output.

4.1 Determination of NARMAX model

The nonlinear system was modeled first for the small amplitude input, and the results are listed in Table 2. An adequate order of the model was 2 and it will be used at the following procedure. To check the capability of the model to represent the linear dynamic properties, modal parameters were calculated from the model and the results are listed in Table 3 with the analytic ones which was obtained by neglecting the effect of the term $k'y^3$, which shows quite coincidence. This means that the nonlinear effects at this input amplitude can be neglected. Table 4 displays two models which are estimated at different input amplitudes, respectively. The first input level was a half of the selected input level by rms magnitude, and the other is selected as one and a half of it. Though there exist small changes in the parameters as the change of input amplitude, the differences fell within the 95% confidence interval of the parameter as listed in Table 2. This shows that the system behaves linearly around at the selected input amplitude.

Table 2 Linear model for a small amplitude input ($\sigma_{\text{Input}}=0.264$ N, $\sigma_{\text{output}}=0.00855$ m)

Order i	Variable a_i	Coefficient θ_i
1	y_{t-1}	1.76386 $(\pm 2.20\times 10^{-5})$
2	y_{t-2}	-0.90437 $(\pm 2.19\times 10^{-5})$
3	u_t	$4.03445\times 10^{-4}(\pm 9.87\times 10^{-8})$
4	u_{t-1}	$1.56232\times 10^{-3}(\pm 9.87\times 10^{-8})$
5	u_{t-2}	$3.83722\times 10^{-4}(\pm 1.06\times 10^{-7})$

Table 3 Comparisons of linear characterisitcs by the NARMAX model estimated by the proposed method with the analytic ones

	Analytic value	NARMAX model by proposed method
Natural frequency	1.233 Hz	1.233 Hz
Damping ratio	0.219	0.219

Table 4 Consistency of model parameters around at the selected input level

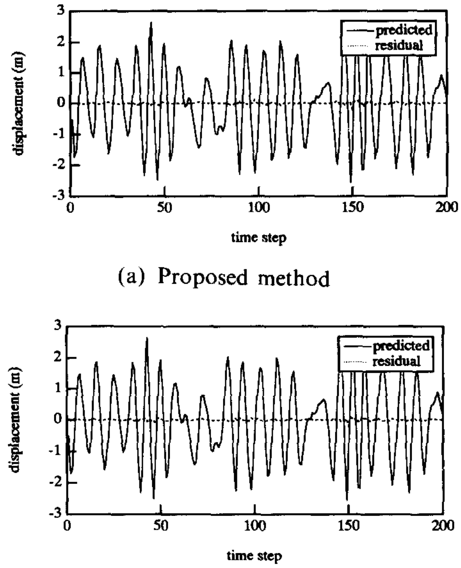
Order I	Variable a_i	Coefficient θ_i	
		$\sigma_{\text{Input}}=0.132$ N	$\sigma_{\text{Input}}=0.396$ N
1	y_{t-1}	1.76387	1.76387
2	y_{t-2}	-0.904837	-0.904837
3	u_t	4.03443×10^{-4}	4.0344810^{-4}
4	u_{t-1}	1.5623210^{-3}	1.5623210^{-3}
5	u_{t-2}	3.8372010^{-4}	3.8372410^{-4}

Table 5 Two nonlinear models for a simulation model estimated by two different methods for a large amplitude input ($\sigma_{input}=146$ N, $\sigma_{output}=1.100$ m)

Order	NARMAX model by the presented method		NARMAX model by the conventional method	
i	Variable a_i	Coefficient θ_i	Variable a_i	Coefficient θ_i
1	y_{k-1}	1.7639	y_{k-1}	1.7336
2	y_{k-2}	−0.9048	y_{k-2}	−0.8933
3	u_k	4.034×10^{-4}	u_k	4.133×10^{-4}
4	u_{k-1}	1.562×10^{-3}	u_{k-1}	1.465×10^{-3}
5	u_{k-2}	3.837×10^{-4}	u_{k-2}	4.164×10^{-4}
6	y^3_{k-1}	−0.1998	y^3_{k-1}	−1.1690
7	y^3_{k-2}	0.0215	y^3_{k-2}	0.0121
8	$y^2_{k-1}y_{k-2}$	0.2089	$y^2_{k-1}y_{k-2}$	0.1395
9	$y_{k-1}y^2_{k-2}$	−0.1564	$y_{k-1}y^2_{k-2}$	−0.1052
1	e_{k-1}	0.00328	e_{k-1}	0.0138
1	e_{k-2}	−0.00238		
$\sqrt{\frac{\Sigma e^2}{\Sigma y^2}}$	0.58%		0.45%	

A nonlinear modeling for the high level by using the proposed method is followed. Nonlinear model terms of Eq. (6) with the maximum time lag 2, which had been determined at the previous stage, were considered in the order of error reduction ratio(ERR). Only the order of nonlinearity was increased until there exists no meaningful term and the model satisfy the unbiasness condition.

Two nonlinear models are presented in Table 5, the first one was constructed by the presented method and the second by the conventional method(Billings and Tsang, 1989a) with the thresholds of 0.01% for ERR and the 99.9% for sum of ERRs. Both two models were composed of with the same number of model terms and model variables, and predict the responses well for the high level input as shown in Fig. 3. In Fig. 4 are shown the correlation tests for the two models, which disply the unbiasness. The residual sum of squareds by the conventional method is smaller than that by the presented method, which are



(a) Proposed method

(b) Conventional method

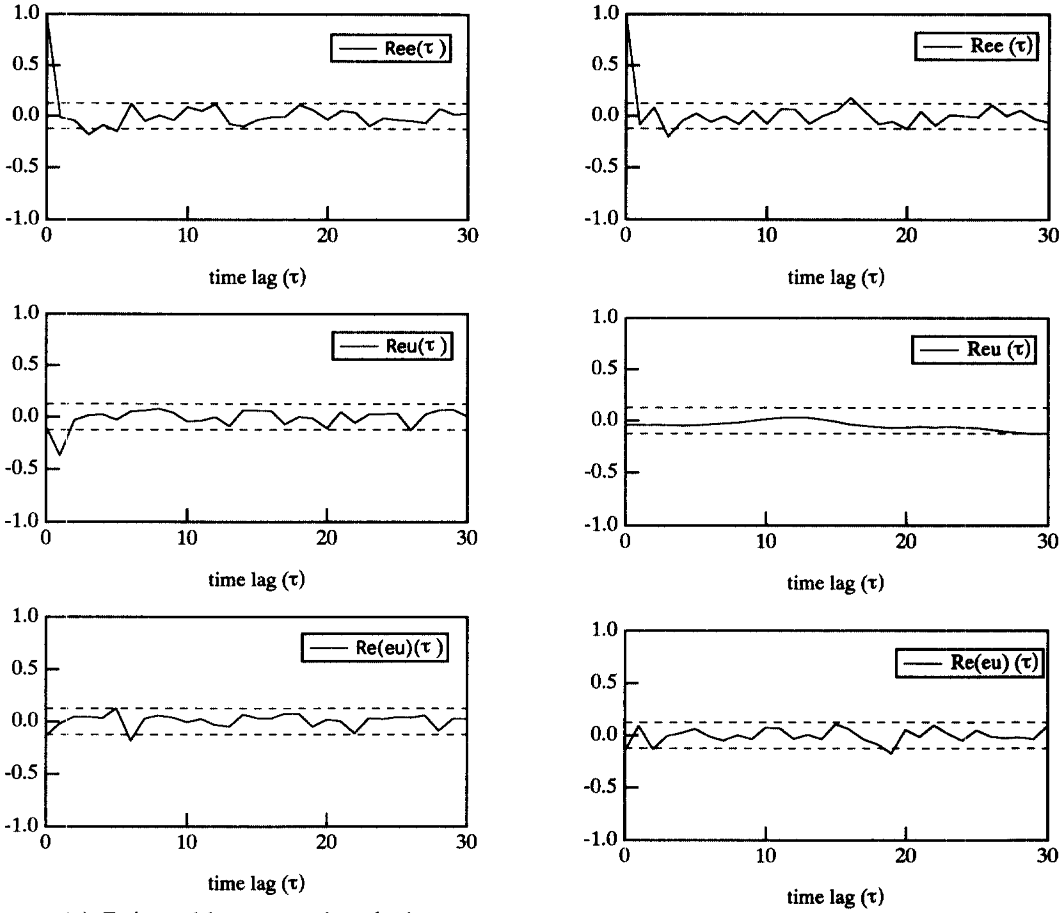
Fig. 3 Predictions of nonlinear responses using NARMAX models estimated by two different modeling techniques respectively

listed in Table 5. As far as only the statistics are concerned, the model by the conventional method seems the better.

4.2 Analysis of nonlinear behaviors in frequency domain

The generalized FRFs of order 1 and 3(H_1 and H_3) extracted from the two kinds of NARMAX models are illustrated in Fig. 5 and 6 with the analytic results of Eq. (3), where H_3 is displayed by contour plots for $f_3=f_1$. Second order FRF does not exist because there are no terms of polynomial order 2. In each figure, three kinds of FRFs are illustrated. The second and third one in Fig. 5 were obtained from the two sets of NARMAX models which are estimated by the presented method and by the conventional method(Billings and Tsang, 1989a) respectively.

H_1 by the presented method shows a typical shape of an FRF for a single dof mechanical system. The peak frequency corresponding to the resonance was 1.233 Hz and the damping ratio was 0.129, which are the same with the analytic one. Therefore, it can be confirmed that the linear properties of the system are kept in the linear part



(a) Estimated by proposed method

(b) Estimated by conventional method

Fig. 4 Correlations validating NARMAX models(- - - : 95% confidence interval)

of the NARMAX model. But, the FRF by the conventional method shows slight differences in natural frequency and damping ratio within 10% respectively, which are 1.31 Hz and 0.135. These are caused by the modeling procedure where the errors are distributed into the whole range of amplitude. In the viewpoint of minimizing the errors, it is not necessary that the linear properties by the estimated nonlinear model should be the same with those by the linearized model. As the linear characteristics do important role in describing nonlinear dynamics as stated previously, the model by the presented method is believed to describe even nonlinear behaviors more accurately.

H_3 by the presented method in Fig. 6, which

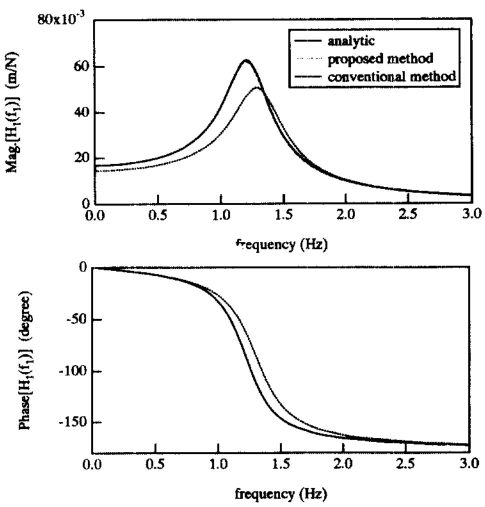
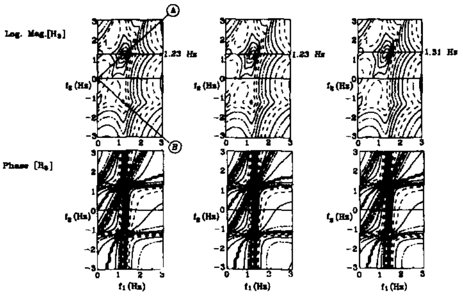


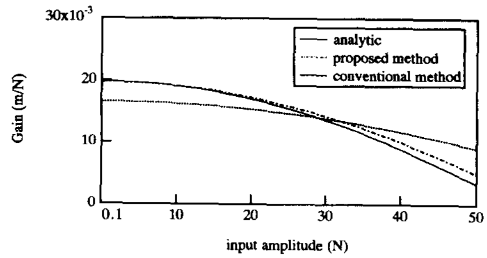
Fig. 5 Comparison of first order generalized FRFs by two different methods with analytic one



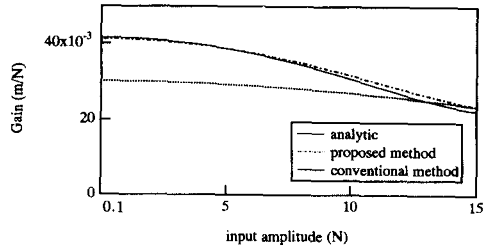
(a) (b) (c)

(a) Analytic FRF
(b) Derived from the model estimated by the proposed method
(c) Derived from the model estimated by the conventional method

Fig. 6 Comparision of third order generalized FRFs by two different methods with analytic one



(a) At a fixed frequency of 0.5 Hz



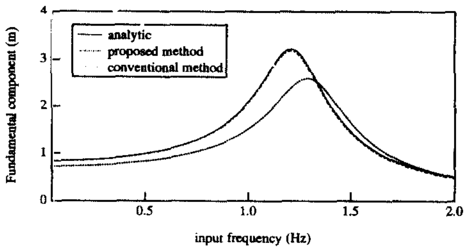
(b) At a fixed frequency of 1 Hz

Fig. 7 Comparison of gain variations at a fixed frequency by two different methods with analytic ones

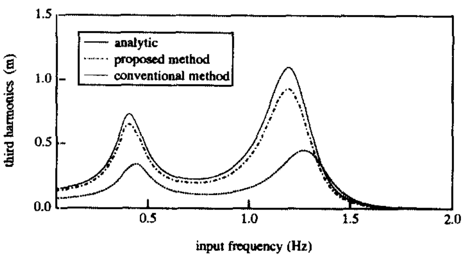
shows almost same results with the analytic FRF, displays two peaks at (1.23, 1.23, 1.23) Hz and at (1.23, −1.23, 1.23) Hz, which mean third harmonics and gain variations at those frequencies respectively. Large magnitudes in the H_3 mean distinct nonlinearities at the corresponding frequencies. Magnitudes of H_3 along the lines $A(f_1, f_1, f_1)$ and $B(f_1, -f_1, f_1)$ in the contour plot represent the 3rd harmonics ($f_o=3f_1$) and the gain variation ($f_o=f_1$) respectively(Billings and Tsang, 1989b). Though H_3 by the conventional method shows similar shapes with the analytic one, the peak frequency and the magnitude were slightly different.

From now on, in order to confirm the capabilities of the proposed procedure in nonlinearity identification, specific nonlinearities such as gain variations and higher harmonics calculated by the two NARMAX models will be compared with the analytic ones. In Fig. 7 are shown the gain variations by the NARMAX models and by the analytic results at fixed frequencies of 0.5 Hz and 1 Hz with the increase of the input amplitude. The results using the FRFs from the NARMAX model by the presented method and the analytical ones show coincident decreases in the gain with the increase of the input, which shows the hardening characteristics of the system. But the model by the conventional method yields somewhat different results in their quantities.

The fundamental components and the third harmonics, obtained by changing the excitation frequency from 0.1 Hz to 2 Hz at fixed amplitude of 50 N, are shown in Fig. 8. The dash-dot lines display the fundamental and third harmonic components by using H_1 and H_3 obtained from the NARMAX model by the presented method, and the solid lines direct calculations by the analytic FRFs. Two kinds of lines display large magnitude at peak frequency of H_1 and at one third of that, respectively, which are well known characteristics of a Duffing's oscillator(Nayfeh and Mook, 1979). Although there exist some discrepancies between the two kinds of results, the maximum difference is about 10%. But the results by the conventional method, illustrated by dotted lines, show distinct differences in the peak frequency and magnitude. General trends for two kinds of nonlinear behaviors, however, by the two techniques are quite similar. That is, the 3rd harmonics around at the peak frequency of H_1 and at one third of the frequency are dominant, and so do the gain virations along the line(f, $-f$, f).



(a) Fundamental components



(b) Third harmonic components

Fig. 8 Comparisons of fundamental and third hamonic components at a constant input amplitude of 50 N by the two different methods with analytical one

The conclusions from the investigations of dominant nonlinearities of the system under identification by two different modeling techniques are as follows. In the viewpoint of the minimizing the residuals, the model by the conventional method is the better. However, the model by the presented method, which puts an emphasis on the linear characteris, describes several nonlinear behaviors more accurately than that by the conventional method.

5. Conclusions

A procedure of estimating an NARMAX model is presented, which enables the better descriptions of nonlinearities in the frequency domain and in the amplitude domain. To resolve the problem of multiplicity of NARMAX models when using the least squares estimators, in the paper, the most adequate model was selected so that the linear characteristics of the system be kept in the nonlinear model. The model is estimated by two steps. At the first step, a linear ARMAX model is derived for low level input, where the degree of freedom of the system, i.e. maximum time lags of the model, as well as linear model structure are obtained. At the second step, since the linear model is taken as the linear part of the NARMAX model, the NARMAX model is estimated by formulating and selecting only the nonlinear terms for high level input.

As an illustration, two NARMAX models were estimated by the method proposed in this paper and by the conventional method, respectively, for a single degree of freedom system with hardening stiffness, and their capabilities to describe the several nonlinearities are compared with the analytic ones. Judging from the results, it is shown that the model by the proposed method yields more accurate descriptions of the nonlinear behaviors of the system.

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